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STEADY-STATE HEAT CONDUCTION FOR A REGION BOUNDED BY A SPHERE AND A TANGENT PLANE

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UDC 536.24.01:517.946

It is shown that the problem of potential theory for a half-space with a spherical cavity with boundary conditions of the first and third kinds reduces to an ordinary differential equation which can be solved efficiently by numerical methods.

It is well known that boundary conditions of the third kind prevent the separation of variables in the general case for boundary-value problems of potential theory. However, as shown in [1, 2], bipolar coordinates in a plane can be used to solve certain problems involving off-center cylinders with a boundary condition of the third kind on the surface of one of the cylinders. In the case of contacting spheres, a system of degenerate bispherical coordinates can be used [3], in which the Fourier–Bessel integral transform method reduces the problem to an ordinary differential equation for the transform. We consider a similar case, when one of the spheres becomes a half-space.

Statement of the Problem. We consider the steady-state temperature distribution between a sphere and a tangent plane with the boundary conditions such that the sphere is at a given constant temperature and the plane is cooled according to Newton's law by a medium at zero temperature (Fig. 1).

In a system of degenerate bispherical corodinates (α, β, φ) related to cylindrical coorcinates (z, ρ, φ) by

$$z + i\rho = \frac{2Ri}{\alpha + i\beta}, \tag{1}$$

$$0 \leqslant \beta \leqslant 1, \quad 0 \leqslant \alpha \leqslant \infty, \quad -\pi \leqslant \varphi \leqslant \pi,$$

the equation of a sphere of radius R becomes $\beta = 1$, and the equation of the tangent plane will be $\beta = 0$. For the case of rotational symmetry, the problem reduces to the solution of Laplace's equation in the form

$$\frac{\partial}{\partial\alpha}\left(\frac{\alpha}{\alpha^2+\beta^2}\frac{\partial T}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{\alpha}{\alpha^2+\beta^2}\frac{\partial T}{\partial\beta}\right) = 0, \quad 0 < \beta < 1, \quad 0 \leqslant \alpha < \infty, \tag{2}$$

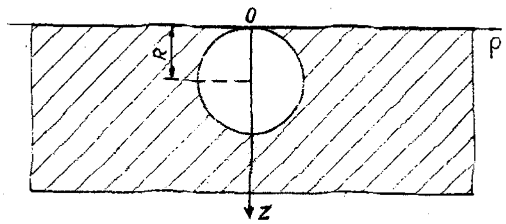

Fig. 1. Half-space with a spherical cavity.

F. Engels Institute of Soviet Commerce, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 47, No. 6, pp. 1006-1010, December, 1984. Original article submitted July 11, 1983.

 0022-0841/84/4706-1478$08.50 © 1985 Plenum Publishing Corporation

subject to the boundary condition on the plane

$$-\alpha^2 \frac{\partial T}{\partial \beta} + HT|_{\beta=0} = 0, \quad 0 \leqslant \alpha < \infty, \tag{3}$$

the condition that the temperature be a constant on the sphere

$$T|_{\beta=1} = T_0, \quad 0 \leqslant \alpha \leqslant \infty, \tag{4}$$

and the condition at infinity

$$\lim_{\sqrt{\alpha^2+\beta^2} \to 0} T(\alpha;\ \beta) = 0, \tag{5}$$

where $T = T(\alpha;\ \beta)$ is the temperature, T_0 is the given constant temperature on the sphere, $H = 2hR$ is the Biot number, and h is a positive constant.

We look for a solution of (2) through (5) in the form [3]

$$T = T_1 + T_2, \tag{6}$$

where

$$T_1 = T_0 \sqrt{\alpha^2 + \beta^2} \int_0^\infty \exp(-x) \frac{\operatorname{ch} \beta x}{\operatorname{ch} x} J_0(\alpha x)\, dx;$$

$$T_2 = T_0 \sqrt{\alpha^2 + \beta^2} \int_0^\infty y(x) \frac{\operatorname{sh}(\beta - 1) x}{\operatorname{ch} x} J_0(\alpha x)\, dx;$$

and $J_0(\alpha x)$ is Bessel's function of order zero.

Substituting (6) in boundary condition (3), we obtain an integral equation for the unknown function y(x):

$$\int_0^\infty y(x)\, [\alpha^2 x + H \operatorname{th} x]\, J_0(\alpha x)\, dx = H \int_0^\infty \frac{\exp(-x)}{\operatorname{ch} x} J_0(\alpha x)\, dx, \ 0 < \alpha < \infty. \tag{7}$$

The integral equation (7) can be transformed to an ordinary differential equation with the help of the identity [4]

$$\alpha^2 \int_0^\infty x y(x)\, J_0(\alpha x)\, dx = -\int_0^\infty \frac{d}{dx}\left(x \frac{dy}{dx}\right) J_0(\alpha x)\, dx, \quad 0 < \alpha < \infty. \tag{8}$$

This identity is true for any continuous, twice-differentiable function on the interval $(0, \infty)$ which is bounded and continuous at the point $x = 0$ and which decays sufficiently rapidly at infinity. Substituting (8) into (7), we have

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) = Hy \operatorname{th} x - H \frac{\exp(-x)}{\operatorname{ch} x}, \quad 0 < x < \infty, \tag{9}$$

subject to the conditions

$$y(x) = O(1), \quad x \to 0, \quad \lim_{x \to \infty} y(x) = 0. \tag{10}$$

The solution of (9) can be found by numerical methods, but it is necessary to transform (10) to an initial condition. This can be done using the general theory of second-order linear differential equations [5].

Solution of the Ordinary Differential Equation. We consider the solution of the corresponding homogeneous equation

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) = Hy \operatorname{th} x, \quad 0 < x < \infty. \tag{11}$$

The point $x = 0$ is a regular singular point of (11). A particular solution of (11) can then be written in series form [5]

$$y_1(x) = \sum_{n=1}^\infty a_n x^{2n}, \quad |x| < \frac{\pi}{2}. \tag{12}$$

The coefficients $\{a_n\}$ are determined successively from the recurrence relation

$$4n^2 a_n = \sum_{k=1}^n q_k a_{n-k}, \quad n = 1,\ 2,\ \ldots, \tag{13}$$

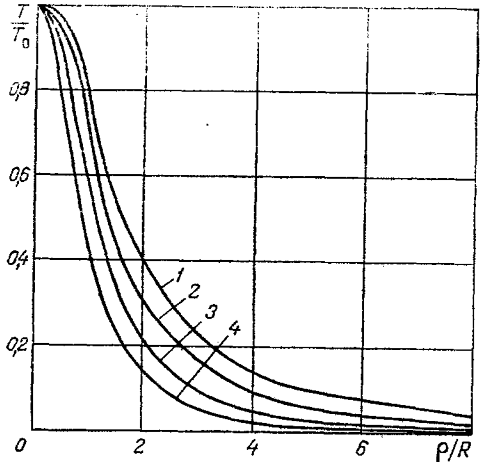

Fig. 2. Temperature distribution on the plane z = 0 (T_0 is the temperature on the surface of the sphere): 1) H = 0.5; 2) H = 1.0; 3) H = 2.0; 4) H = 4.0.

where

$$q_k = H\,\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!};$$

with $a_0 = 1$, and B_{2k} are the Bernoulli numbers.

Solution (12) also satisfies a Volterra integral equation [5]

$$y_1(x) = 1 + H\int_0^x y_1(t)\,\mathrm{th}\,t\ln\left(\frac{x}{t}\right)dt,\quad 0\leqslant x<\infty. \tag{14}$$

Equation (14) can be solved numerically or by the method of successive approximations for $x \gg \pi/2$. It also follows from (14) that (12) monotonically increases on $(0, \infty)$ and has the asymptotic form

$$y_1(x) = O\{x^{-\frac{1}{4}}\exp(2\sqrt{Hx})\},\quad x\to\infty. \tag{15}$$

The numerical solution of (12) on $(0, \infty)$ can be done using the method of Adams and Shtërmer [5]. If a solution $y_1(x)$ is known, then a second solution $y_2(x)$ can be found by quadratures from the condition [4]

$$W(y_1;\, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{1}{x}, \tag{16}$$

where $W(y_1;\, y_2)$ is the Wronskian determinant.

The solution $y_2(x)$ can be written in the form

$$y_2(x) = y_1(x)\int_x^\infty \frac{dt}{t y_1^2(t)},\quad 0<x<\infty. \tag{17}$$

Solution (17) has the following asymptotic forms for large and small values of the argument x:

$$\begin{gathered} y_2(x) = O\{x^{-\frac{1}{4}}\exp(-2\sqrt{Hx})\},\quad x\to\infty, \\ y_2(x) = O\left(\ln\frac{1}{x}\right),\quad x\to 0. \end{gathered} \tag{18}$$

Using the method of variation of parameters, we obtain a particular solution of the nonhomogeneous equation (9) in the form

$$y(x) = y_1(x)\int_x^\infty \frac{H\exp(-t)}{\mathrm{ch}\,t}\,y_2(t)\,dt + y_2(x)\int_0^x \frac{H\exp(-t)}{\mathrm{ch}\,t}\,y_1(t)\,dt,\ 0<x<\infty. \tag{19}$$

From (15) and (18) we have that (19) satisfies condition (10), while (12) and (17) do not satisfy this condition; hence (19) in the unique solution. Also from (12), (15), and (18) it follows that (19) satisfies the more stringent conditions that the identity (8) be applicable.

As an example, in Fig. 2 we show the temperature distribution on the plane $z = 0$ for $H \in \{0.5, 1, 2, 4\}$. The calculations were done with the Adams and Shtërmer method on the ES-1045 computer.

Finally, similar results can be obtained for the case when the boundary condition (4) depends on the angle φ.

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